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Rational connectedness and Galois covers of the projective line

By Jean-Louis Colliot-Thélène

Let k be a p-adic field. Some time ago, D. Harbater [9] proved that any finite group G may be realized as a regular Galois group over the rational function field in one variable $k(t)$, namely there exists a finite field extension $F/k(t)$, Galois with group G, such that F is a regular extension of k (i.e. k is algebraically closed in F). Moreover, one may arrange that a given k-place of $k(t)$ be totally split in F. Harbater proved this theorem for k an arbitrary complete valued field. Rather formal arguments ([10, §4.5]; §2 hereafter) then imply that the theorem holds over any 'large' field k. This in turn is a special case of a result of Pop [15], hence will be referred to as the Harbater/Pop theorem. We refer to [10], [16], [6] for precise references to the literature (work of Dèbes, Deschamps, Fried, Haran, Harbater, Jarden, Liu, Pop, Serre, and Völklein).

Most proofs (see [10], [19, 8.4.4, p. 93] and Liu's contribution to [16]; see however [15]) first use direct arguments to establish the theorem when G is a cyclic group (here the nature of the ground field is irrelevant), then proceed by patching, using either formal or rigid geometry, together with GAGA theorems.

In the present paper, where I take the case of algebraically closed fields for granted, I show how a technique recently developed by Kollár [12] may be used to give a quite different proof of the Harbater/Pop theorem, when the 'large' field k has characteristic zero. This proof actually gives more than the original result (see comment after statement of Theorem 1).

Before I formally state the main result, let us recall what a 'large' field is. Let k be a field and let $k((y))$ be the quotient field of the ring $k[[y]]$ of formal power series in one variable. Following F. Pop, we shall say that k is 'large' if it satisfies one of the three equivalent properties ([15, Prop. 1.1]):

(i) It is existentially closed in $k((y))$: any k-variety with a $k((y))$-point has a k-point.

(ii) On a smooth integral k-variety with a k-point, k-points are Zariski dense.

(iii) On a smooth integral k-curve with a k-point, k-points are Zariski dense.

(Such a field is clearly infinite. By going over to the completion at a smooth k-point of a curve, one sees that (i) implies (iii). That (iii) implies (ii) is easy (consider a regular system of parameters). In characteristic zero, one may use resolution of singularities to show that (ii) implies (i).)

Known examples of 'large' fields k are fraction fields of a henselian discrete valuation ring, such as a p-adic field or a field of the shape $k = F((x))$ for F some field.

Other well-known examples are real closed fields. That these are 'large' is a special instance of the following fact, which seems to have escaped the attention of specialists: any field F, all finite field extensions of which are of degree a power of a fixed prime p, is a 'large' field. To see this, one only needs to observe that on a regular, projective, connected curve C over a field F, given any nonempty open set U, any zero-cycle (divisor) z on C is rationally equivalent to a zero-cycle z_1 whose support is contained in U (a semi-local Dedekind ring is a principal ideal domain); the degree (over F) of z and z_1 clearly coincide. Applying this to an F-point of C, one produces a zero-cycle $\sum_i n_i P_i$ ($n_i \in \mathbf{Z}$, P_i closed points) with support in U, such that the degree $\sum_i n_i [F(P_i) : F] = 1$. For F as above, this forces one of the degrees $[F(P_i) : F]$ to be one.

Other known examples are the fields of totally real algebraic numbers and of totally p-adic algebraic numbers (that these fields are 'large' is a very special case of a theorem of Moret-Bailly [14, Thm. 1.3]). The property trivially holds for so-called pseudo algebraically closed fields, such as infinite algebraic extensions of a finite field.

THEOREM 1. *Let G be a finite group. Let k be a 'large' field of characteristic zero. Let $\mathcal{E} = \mathrm{Spec}(K)$ be a G-torsor over $\mathrm{Spec}(k)$. Then there exist an open set U of the affine line $\mathbf{A}^1_k$ containing a k-point O and a G-torsor $V \to U$ such that the following two properties hold*:

(i) *The fibre of $V \to U$ over O is isomorphic to $\mathcal{E}$ (as a G-torsor over* $\mathrm{Spec}(k)$);

(ii) *The smooth k-curve V is geometrically connected.*

The ring K is a finite separable extension of k; it need not be a field. In loose terms: given a Galois extension K/k with group G, one may realize G as the Galois group of a 'regular' extension of $k(t)$, in such a way that over a suitable k-place of $k(t)$, the extension specializes to K/k.

When the G-torsor $\mathcal{E}/\mathrm{Spec}(k)$ is trivial, i.e. $\mathcal{E} = \coprod_{g \in G} \mathrm{Spec}(k)$, we recover the result of Harbater and Pop. The question whether $\mathcal{E}$ may be chosen arbitrary had been investigated for special groups by several authors (see [6]). For arbitrary groups, Dèbes proves a weaker result ([6, Thm. 3.1]) when k is

'large', and he proves the theorem in the case where k is a pseudo algebraically closed field ([6, Thm. 3.2]).

Using general results from [EGA IV_3], we immediately obtain a series of concrete corollaries. These will be detailed in Section 2. In the case of a split $\mathcal{E}/k$, most of them had already been obtained, with somewhat different proofs.

After the paper was submitted, I was asked whether in Theorem 1 one may impose arbitrary G-torsors as fibres of $V \to U$ at more than one k-point of $U \subset \mathbf{A}^1_k$. The answer is in general in the negative, as shown in the appendix.

Let us say a few words on the tools used in this article. In a series of papers which appeared in 1992, Kollár, Miyaoka and Mori developed a technique which enables them, under some assumptions, to smooth a tree of rational curves into a single rational curve ([13, Thm. (2.1)]; see also [11, Chap. II. 7, pp. 154–158] and [5, §4.2]). That work was over an algebraically closed field. In his recent paper [12], Kollár extends the technique over 'large' fields (e.g. local fields). Under certain assumptions, he manages to deform a set of conjugate $\mathbf{P}^1$'s into a single $\mathbf{P}^1$ defined over the ground field. From this he gets the finiteness of the set of R-equivalence classes on k-points of a geometrically rationally connected variety defined over a local field k. That the key lemma of [12] precisely holds for 'large' fields provided the incentive for the present paper.

The proof I give for Theorem 1 starts from the classical fact that a finite group G is a Galois group over $k(t)$ when k is algebraically closed of characteristic zero. It then uses a natural versal model for a G-torsor, and applies the deformation result of [12] to (a smooth compactification of) the base space of this G-torsor. The proof uses the existence of such a smooth compactification, but it avoids any consideration of the divisor at infinity: there is no discussion of inertia groups at all.

The idea of using a versal model of a G-torsor, originally due to E. Noether, has come up a number of times in the literature, notably in work of E. Fischer, D. Saltman [17], F. A. Bogomolov [1]; see [20] and [21] for further references.

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1. Proof of Theorem 1

In this section, we shall assume that the ground field k (which is of characteristic zero) is uncountable. The proof in the countable case will be given in Section 2.

Let $\overline{k}$ be an algebraic closure of k. Given a k-scheme Z, let us write $\overline{Z} = Z \times_k \overline{k}$.

(1) Let G be a finite group and $\mathcal{E}/\mathrm{Spec}(k)$ a G-torsor. Let us fix an embedding of G into some general linear group GL_n. Here G is viewed as a constant (split) k-group scheme, GL_n is the linear group over k and $i : G \to \mathrm{GL}_n$ is a homomorphism of k-group schemes. Let $U = \mathrm{GL}_n/G$ be the affine k-variety of 'left classes'. This is the affine k-scheme whose ring is the ring of invariants for G acting on the ring $k[\mathrm{GL}_n]$. The projection map $\mathrm{GL}_n \to U$ makes GL_n into a right G-torsor V over U. The left action of GL_n on itself induces a left action of GL_n on $U = \mathrm{GL}_n/G$ and the projection $V \to U$ is equivariant for these (left) actions.

Let us recall basic facts from noncommutative étale cohomology. Given any smooth affine k-group scheme H, and any commutative k-algebra A, we denote by $H^1_{\text{ét}}(A, H)$ the pointed cohomology set which classifies (étale) (right) $H \times_k A$-torsors over $\mathrm{Spec}(A)$ (up to nonunique isomorphism). Such torsors will simply be called H-torsors over A. For any such A, there is an "exact sequence"

$$V(A) \to U(A) \to H^1_{\text{ét}}(A, G) \to H^1_{\text{ét}}(A, \mathrm{GL}_n).$$

Let us detail this sequence. The map $V(A) \to U(A)$ is the obvious one; it respects the (left) action of $\mathrm{GL}_n(A)$ on both sets. The right G-torsor $V \to U$ defines an element $\xi \in H^1_{\text{ét}}(U, G)$. To an element $\rho \in U(A) = \mathrm{Hom}_k(\mathrm{Spec}(A), U)$, the map $U(A) \to H^1_{\text{ét}}(A, G)$ associates the class $\rho^*(\xi) \in H^1_{\text{ét}}(A, G)$ of the pull-back $\rho^*(V \to U)$, which is a G-torsor over A. Two points $x, y \in U(A)$ have the same image in $H^1_{\text{ét}}(A, G)$ if and only if there exists $\alpha \in \mathrm{GL}_n(A)$ such that $\alpha.x = y$. By Grothendieck's version of Hilbert's Theorem 90, the set $H^1_{\text{ét}}(A, \mathrm{GL}_n)$ classifies projective modules of rank n over A. Thus if A is semi-local, or if A is a Dedekind ring with trival class group, then $H^1_{\text{ét}}(A, \mathrm{GL}_n)$ is reduced to one element, and for any right G-torsor $\mathcal{T}$ over A there exists an element $\rho \in U(A)$ such that $\mathcal{T}$ and $\rho^*(V \to U)$ are isomorphic G-torsors over A. In particular, there exists a k-point $P \in U(k)$ such that the fibre V_P of V above P is a G-torsor isomorphic to the given $\mathcal{E}/k$. We shall fix such a k-point P.

(2) By classical results (see [19, Chap. 6]), we know that G is a 'regular' Galois group over $\overline{k}(t)$. In other words there exist a nonempty open set W of the affine line $\mathbf{A}^1_{\overline{k}} = \mathrm{Spec}(\overline{k}[t])$ and a G-torsor over W whose underlying variety is integral. Let A be the semi-local ring of $\overline{k}[t]$ at $t = 0$ and $t = 1$, and let $S = \mathrm{Spec}(A)$. Let us abuse notation and call 0, respectively 1, the points of S defined by $t = 0$, respectively $t = 1$. Changing coordinates and semi-localizing produces a G-torsor $\mathcal{T}$ over S such that $\mathcal{T}$ is an integral scheme.

By (1), there exists a nonconstant $\overline{k}$-morphism $\rho : S \to \overline{U}$ such that the pull-back of the G-torsor $\overline{V} \to \overline{U}$ under ρ is isomorphic to the G-torsor $\mathcal{T}/S$. Given any $\alpha \in \mathrm{GL}_n(A)$, the G-torsor $(\alpha.\rho)^*(\overline{V} \to \overline{U})$ is G-isomorphic to the G-torsor $\mathcal{T}$. In particular, it is an integral scheme.

(3) The action of $\mathrm{GL}_n(\overline{k})$ on $\overline{U}(\overline{k})$ is transitive; hence the obvious action of $\mathrm{GL}_n(\overline{k}) \times \mathrm{GL}_n(\overline{k})$ on $\overline{U}(\overline{k}) \times \overline{U}(\overline{k})$ is also transitive. Reduction of A modulo t and modulo $t-1$ induces a surjective homomorphism $\mathrm{GL}_n(A) \to \mathrm{GL}_n(\overline{k}) \times \mathrm{GL}_n(\overline{k})$. Thus given two points $M, N \in \overline{U}(\overline{k})$, there exists $\alpha \in \mathrm{GL}_n(A)$ such that $\alpha.\rho \in \overline{U}(A)$ sends the point $t = 0$ to M and the point $t = 1$ to N.

Remark. One should compare the present general position argument with 'Kuyk's lemma' (see [20, Lemma 4.5]).

(4) Since char(k)=0, by Hironaka's theorem, there exist smooth, projective, geometrically integral k-varieties X_1 and X, with V open in X_1 and U open in X, together with a k-morphism $p : X_1 \to X$ extending the map $V \to U$ and inducing a k-isomorphism $V \simeq p^{-1}(U)$.

(5) According to a theorem of Kollár, Miyaoka and Mori ([13]; [11, Thm. II. 3.11, p. 118]), to the point $\overline{P} \in \overline{U}(\overline{k}) \subset \overline{X}(\overline{k})$ one may associate countably many proper subvarieties V_i ($i \in I$) of the smooth projective variety $\overline{X}$ such that if $f : \mathbf{P}^1_{\overline{k}} \to \overline{X}$ is a nonconstant morphism, $f(0) = \overline{P}$ and the image of f is not contained in the union of the V_i's, then f is free over $0 \in \mathbf{P}^1_{\overline{k}}$. By definition (see [11, II. 3.1, p. 113]), this means that the coherent cohomology group $H^1(\mathbf{P}^1_{\overline{k}}, f^*T_{\overline{X}}(-2))$ vanishes (here $T_{\overline{X}}$ denotes the tangent bundle of $\overline{X}$), which amounts to the hypothesis that in Grothendieck's decomposition of the vector bundle $f^*T_{\overline{X}}$ over $\mathbf{P}^1_{\overline{k}}$ as a sum of line bundles $\mathcal{O}_{\mathbf{P}^1}(n_j)$, we have $n_j > 0$ for each j (this is the ampleness property for the vector bundle $f^*T_{\overline{X}}$ on $\mathbf{P}^1_{\overline{k}}$, see [11, II.3.8, p. 116]).

Since k is uncountable, there exists a point $Q \in \overline{U}(\overline{k})$, $Q \neq \overline{P}$, which does not lie on any of the V_i's (proof: use a generically finite projection to projective space and induct on dimension). By (3), there exists $\alpha \in \mathrm{GL}_n(A)$ such that $\alpha.\rho \in \overline{U}(A)$ sends the point $t = 0$ to $\overline{P}$ and the point $t = 1$ to Q. Since X/k is proper, the morphism $\alpha.\rho : S \to \overline{U}$ extends to a (nonconstant) morphism $f : \mathbf{P}^1_{\overline{k}} \to \overline{X}$. The image of f contains $\overline{P}$ and is not contained in the union of the V_i's, since this image contains Q. By the quoted theorem ([11, II.3.11]), we conclude:

(5.1) *The vector bundle $f^*T_{\overline{X}}$ on $\mathbf{P}^1_{\overline{k}}$ is ample.*

On the other hand, we have:

(5.2) *The underlying variety of the G-torsor $f^*(\overline{V} \to \overline{U})$ over $f^{-1}(\overline{U})$ is integral.*

Indeed, this follows from the same statement for the restriction of this G-torsor over $S = \mathrm{Spec}(A) \subset f^{-1}(\overline{U})$, which was pointed out at the end of (2).

(6) We have now reached the situation studied in [12]. Starting from $f : \mathbf{P}^1_{\overline{k}} \to \overline{X}$ such that $f(0) = \overline{P}$ and $f^*T_{\overline{X}}$ is ample, Kollár ([12, 3.2], I change notation) produces, over the ground field k, a smooth integral k-curve C with a k-point O, a smooth geometrically integral k-surface Z proper over C, together with a k-morphism $h : Z \to X$, with the following properties:

(6.a) The projection $Z \to C$ admits a k-section $\sigma : C \to Z$ which by h is mapped to $P \in X$.

(6.b) The geometric fibre $Z_{\overline{O}}$ of $Z \to C$ at the point O is a comb $D + \sum_{i \in I} C_i$ on $\overline{Z}$ (here I is a nonempty finite set, the C_i's are the teeth of the comb, see [11, II.7.7, p. 156]), each component of which is a nonsingular curve of genus zero; the map $\overline{h} : \overline{Z} \to \overline{X}$ sends D to $\overline{P}$ and induces on C_i a conjugate of $f : \mathbf{P}^1_{\overline{k}} \to \overline{X}$.

(6.c) Over any closed point M of C different from O, the fibre Z_M of $Z \to C$ is $k(M)$-isomorphic to the projective line $\mathbf{P}^1_{k(M)}$: the fibre is a smooth, geometrically irreducible, projective curve of genus zero over the residue field $k(M)$, and it contains the $k(M)$-rational point $\sigma(M)$.

(7) Since the map $\overline{h} : Z_{\overline{O}} \to \overline{X}$ is not constant (because its restriction to any C_i is not constant), the closed set $h^{-1}(P) \subset Z$ is a proper closed set. Thus, after shrinking C, we may assume: for no $M \in C$ is h constant on the fibre Z_M (note that on any fibre Z_M, h assumes the value $h(\sigma(M)) = P \times_k k(M)$).

Let $\Omega \subset Z$ be the inverse image of U under h. Note that Ω contains $\sigma(C)$, hence the composite map $\Omega \subset Z \to C$ is surjective. Let $\Omega_1 \to \Omega$ be the inverse image of the G-torsor $V \to U$ under $h : \Omega \to U$. Let M be a closed point in C. We shall show: *For all but finitely many $M \in C$, the total space of the induced G-torsor $\Omega_{1,M} \to \Omega_M \subset Z_M \simeq \mathbf{P}^1_{k(M)}$ is a smooth geometrically integral $k(M)$-variety.*

To prove this, it is enough to prove the corresponding statement over $\overline{k}$. For the rest of the proof of (7), to simplify notation, let us set $k = \overline{k}$. Points M will be $\overline{k}$-rational points on C. For $M \neq O$, the (nonempty) variety Ω_M is smooth and connected and the variety $\Omega_{1,M}$ is a finite étale cover of Ω_M, hence is smooth. To prove that a given $\Omega_{1,M}, M \neq O$, is integral, it is thus enough to show that it is connected.

The inverse image in Ω_1 of $D \cap \Omega$ is a disjoint union of copies D_g $(g \in G)$ of $D \cap \Omega$, each with multiplicity one; by (5.2) and (6.b), for a given $i \in I$ the inverse image in Ω_1 of each $C_i \cap \Omega$ is a (smooth) *connected* curve, which meets *each* D_g $(g \in G)$, since C_i meets D (see (6.b)). Thus $\Omega_{1,O}$, which is the inverse image of $D + \sum_{i \in I} C_i$, is a *reduced connected* divisor on Ω_1.

That $\Omega_{1,M}$ is connected for all but finitely many $M \in C$ now follows from the general lemma (where X and Y have nothing to do with the previous Y and X), to be applied to $X = \Omega_1$ and $Y = \Omega$:

LEMMA. *Let C be a smooth, connected curve over an algebraically closed field k, and let $O \in C(k)$. Let X, Y, C be smooth varieties over k, equipped with faithfully flat k-morphisms $X \to Y$ and $Y \to C$. Assume that the generic fibre of $Y \to C$ is smooth and geometrically integral. Assume that $X \to Y$ is finite and étale. Assume moreover that the inverse image of O under the composite map $X \to Y \to C$ is a connected divisor on X and is not a multiple divisor. Then there exists a finite set S of points of C such that for $M \in C, M \notin S$, the inverse image X_M of M under the composite map $X \to Y \to C$ is a smooth connected variety.*

Proof. Note first that X is connected. Indeed if it was not connected, the finite étale cover $X \to Y$ would break up into a disjoint union of finite étale (hence faithfully flat) covers $X_i \to Y$, and the fibre of $X \to Y \to C$ over O would not be connected. Thus X is connected; since it is smooth, it is integral. Let D be the normalization of C in the function field of X. This is a smooth integral curve, and the map $D \to C$ is flat and finite. Since X is normal, the map $X \to C$ factors through D. The finite (étale) map $X \to Y$ factors through the scheme $Y \times_C D$. The scheme $Y \times_C D$ is integral, because C is its own normalization in Y, since we have assumed that the generic fibre of $Y \to C$ is geometrically integral. The finite map of integral varieties $X \to Y \times_C D$ is dominant, hence surjective as a morphism of schemes (it need not be flat). In particular, it is surjective on k-points (recall $k = \overline{k}$). The projection map $Y \times_C D \to D$ is faithfully flat, since it is obtained by base change from the faithfully flat map $Y \to C$. In particular, $Y \times_C D \to D$ is surjective on k-points. We conclude that $X \to D$ is surjective on k-points. But then the scheme-theoretic inverse image of $O \in C$ under the map $D \to C$ must consist of one reduced point, since the inverse image of O under the composite map $X \to D \to C$ is a connected divisor which is not multiple. Since $D \to C$ is finite and flat, this implies that $D \to C$ is an isomorphism. Thus the function field of C is algebraically closed in the function field of X, hence the generic fibre of $X \to C$ is a smooth geometrically integral variety. By [EGA IV_3, (9.7.7)] this implies the same statement for all fibres of $X \to C$ away from a proper closed subset of C. □

(8) We finally make use of the hypothesis that the field k is 'large.' Since the curve C has a k-rational point, namely O, this hypothesis implies that there exists a k-point M on C away from the finitely many points excluded in (7), such that the map $\mathbf{P}^1_k \to X$ induced by h on the fibre $Z_M \simeq \mathbf{P}^1_k$

does what we want: the inverse image of the G-torsor $V \to U$ under the map $h : h^{-1}(U) \cap \mathbf{P}^1 \to U$ is a G-torsor over the open set $h^{-1}(U) \subset \mathbf{P}^1_k$, whose fibre at $\sigma(M) \in h^{-1}(U)(k) \subset \mathbf{P}^1(k)$ is isomorphic to the fibre of $V \to U$ at P, hence is isomorphic to $\mathcal{E}$ (by the very choice of P, see (1)), and whose total space is a geometrically integral k-variety (see (7)).

2. Corollaries

Theorem 2. *Let O be a $\mathbf{Q}$-point of the projective line $\mathbf{P}^1_{\mathbf{Q}}$. Let G be a finite group and let $\mathcal{E} = \mathrm{Spec}(K) \to \mathrm{Spec}(\mathbf{Q})$ be a G-torsor. There exist a smooth, geometrically integral curve $Y/\mathbf{Q}$ whose smooth compactification has a $\mathbf{Q}$-point, an open set $U \subset \mathbf{P}^1 \times_{\mathbf{Q}} Y$ containing $O \times_{\mathbf{Q}} Y$, and a G-torsor $V \to U$ (an étale Galois cover with group G), whose restriction to $O \times_{\mathbf{Q}} Y$ is the G-torsor $\mathcal{E} \times_{\mathbf{Q}} Y$, and such that the fibre of the composite map $V \to U \to Y$ at any geometric point of Y is nonempty and connected (hence integral).*

Proof. Let $G \hookrightarrow \mathrm{GL}_{n,\mathbf{Q}}$ be an embedding. The varieties U, V, X, X_1 which appear in the proof of Theorem 1 may all be defined over $\mathbf{Q}$. We also have $P \in U(\mathbf{Q}) \subset X(\mathbf{Q})$.

For any field F with $\mathbf{Q} \subset F$, let us in this proof say that an F-morphism $f : \mathbf{P}^1_F \to X_F$ is *good* if $f(O) = P_F$ and the inverse image of $V_F \to U_F$ under f (restricted to $f^{-1}(U_F)$) is a geometrically integral F-variety. Let $Z = \mathrm{Hom}_{\mathbf{Q}}(\mathbf{P}^1, X, O \mapsto P)$ (notation as in [11, II.1.4, p. 94]). This is a countable union of $\mathbf{Q}$-varieties Z_d (d for degree of the image of $\mathbf{P}^1$, in a fixed projective embedding of X). An F-point of Z will be called good if the corresponding F-morphism $f : \mathbf{P}^1_F \to X_F$ is good. Given arbitrary field extensions $\mathbf{Q} \subset E_1 \subset E_2$, a point in $Z(E_1)$ is good if and only if its image in $Z(E_2)$ is good.

The field $\mathbf{Q}((x))$ is uncountable. By Theorem 1 over such a field, as proved in Section 1, there exists a good $\mathbf{Q}((x))$-point on Z, hence on Z_d for some d. Let $Y \subset Z_d$ be the scheme-theoretic closure of the image of the corresponding morphism $\mathrm{Spec}(\mathbf{Q}((x))) \to Z_d$. The $\mathbf{Q}$-variety Y is geometrically integral. We have the field embeddings $\mathbf{Q} \subset \mathbf{Q}(Y) \subset \mathbf{Q}((x))$. Thus on the one hand the generic point of Y is a good $\mathbf{Q}(Y)$-point of Z; on the other hand any $\mathbf{Q}$-compactification of Y has a $\mathbf{Q}$-point. Indeed, for any such compactification Y_c, the map $\mathrm{Spec}(\mathbf{Q}((x))) \to Y$ extends to a $\mathbf{Q}$-morphism $\mathrm{Spec}(\mathbf{Q}[[x]]) \to Y_c$; the image of $x = 0$ is a $\mathbf{Q}$-point of Y_c.

Replacing Y by a nonempty open set, one may ensure ([EGA IV_3, (8.8.2)]) that the corresponding good $\mathbf{Q}(Y)$-morphism $\mathbf{P}^1_{\mathbf{Q}(Y)} \to X_{\mathbf{Q}(Y)}$ extends to a Y-morphism $\varphi : \mathbf{P}^1 \times_{\mathbf{Q}} Y \to X \times_{\mathbf{Q}} Y$ which sends $O \times_{\mathbf{Q}} Y$ to $P \times_{\mathbf{Q}} Y$.

Let $\Omega = \varphi^{-1}(U \times_{\mathbf{Q}} Y) \subset \mathbf{P}^1 \times_{\mathbf{Q}} Y$ and let $\Omega_1 \to \Omega$ be the G-torsor which is the inverse image of the G-torsor $V \times_{\mathbf{Q}} Y \to U \times_{\mathbf{Q}} Y$ under φ. Upon replacing Y by a nonempty open set (this is actually not necessary), the restriction of this G-torsor over $O \times_{\mathbf{Q}} Y \subset \Omega$ is isomorphic to $\mathcal{E} \times_{\mathbf{Q}} Y$ (indeed, this is true over the generic point of Y). We have the maps $\Omega_1 \to \Omega \to Y$. The first map is finite étale of constant rank, the second one is smooth and surjective. Thus the composite map $\Omega_1 \to Y$ is smooth. Since the generic point of Y corresponds to a good point of Z, the generic fibre $\Omega_{1,\mathbf{Q}(Y)}$ is geometrically integral over $\mathbf{Q}(Y)$. Upon replacing Y by a nonempty open set ([EGA IV$_3$, (9.7.7)(iv)]), we therefore have that all geometric fibres of the map $\Omega_1 \to Y$ are smooth and geometrically integral. In particular for any field F with $\mathbf{Q} \subset F$ and any F-point of Y, the morphism $\varphi_F : \mathbf{P}^1_F \to X_F$ induced by φ is good.

On a smooth projective model Y_c of Y over $\mathbf{Q}$, there exists a $\mathbf{Q}$-point R. By considering a regular system of parameters at R one produces a geometrically integral $\mathbf{Q}$-curve $C \subset Y_c$, smooth at R, and which meets Y. One now replaces Y by $Y \cap C$. This completes the proof of Theorem 2. $\square$

Remarks and corollaries.

(1) Note that Y in Theorem 2 need not have a $\mathbf{Q}$-point. But for any field k containing $\mathbf{Q}$ such that $Y(k) \neq \emptyset$, G is a 'regular' Galois group over the rational field $k(t)$, with the added information that the fibre at the point $t = 0$ is isomorphic to the torsor $\mathcal{E} \times_{\mathbf{Q}} k$. This applies in particular to any 'large' field of characteristic zero, thus *completing the proof of Theorem* 1 *for fields which are countable.*

(2) One should compare Theorem 2 with the contribution of Deschamps in [16], and the proof given here with that given in [7, 4.2].

(3) One amusing corollary is that *for any finite group G, there exists a finite set of number fields k_i such that the greatest common denominator of the degrees $[k_i : \mathbf{Q}]$ is equal to one, and such that G is a 'regular' Galois group over each $k_i(t)$, hence in particular a Galois group over each k_i.* The proof is simple: on the smooth compactification Y_c of the curve Y, there exists a $\mathbf{Q}$-point, call it M. If we let $S \subset Y_c$ be the complement of Y in Y_c, there exists a zero-cycle $\sum_{i \in I} n_i P_i$ (here the n_i are integers, P_i is a closed point and I is finite) on Y_c which is rationally equivalent to M, hence of degree one, and whose support is foreign to S, i.e. whose support is contained in Y. Let k_i be the residue field at the closed point P_i. Then $\sum_{i \in I} n_i [k_i : \mathbf{Q}] = 1$ and $Y(k_i) \neq \emptyset$ for each i, hence the claim.

One could say that, for any group G, the inverse Galois group problem over $\mathbf{Q}$ acquires a positive answer when passing from rational points to 'zero-cycles of degree one.'

This could have been noticed earlier. For any prime p, let K_p be the fixed field of a pro-p-Sylow subgroup of the absolute Galois group of $\mathbf{Q}$. As proved in the introduction of this paper, K_p is a 'large' field. By Theorem 1 (or, for that matter, the Harbater/Pop theorem), G is a regular Galois group over $K_p(t)$. There exists a finite subextension $L_p/\mathbf{Q}$ of $K_p/\mathbf{Q}$, such that G is a regular Galois group over $L_p(t)$. By Hilbert's irreducibility theorem, G is a Galois group over the number field L_p, whose degree $[L_p : \mathbf{Q}]$ is prime to p.

(4) Starting from the statement of Theorem 2 and writing a model of the whole situation over an open set of the ring of integers (same references to [EGA IV$_3$] as above), one easily deduces the following result, which is a special case of a theorem of Fried and Völklein: *For a given finite group G, for almost all primes p ("almost all" depending on G), G is a 'regular' Galois group over* $\mathbf{F}_p(t)$ (see [10] and [7, 3.9] for references; in [7] a model-theoretic argument is given). Simply note that if $\mathcal{Y}/\mathbf{Z}$ is a smooth integral model of the smooth, geometrically integral curve $Y/\mathbf{Q}$, then by classical estimates (Weil) we have $\mathcal{Y}(\mathbf{F}_p) \neq \emptyset$ for almost all primes p. Here again, the present proof enables us to get more: if we start off with a given G-torsor $\mathcal{E}$ over a nonempty open set of $\mathrm{Spec}(\mathbf{Z})$, we may satisfy the additional requirement that for almost all primes p the 'regular' Galois extension over $\mathbf{F}_p(t)$ be unramified at $t = 0$, the fibre being isomorphic to $\mathcal{E} \times_{\mathbf{Z}} \mathbf{F}_p$.

Appendix

In this appendix, where for simplicity I assume all fields to be of characteristic zero, I address the question:

Let k be a field, G a finite group, $n \geq 1$ an integer. Let $\mathcal{E}_1, \cdots, \mathcal{E}_n$ be G-torsors over k. Can one find an open set $U \subset \mathbf{A}^1_k$, a G-torsor $V \to U$ and n points $P_1, \cdots, P_n \in U(k)$ such that for each i, the fibre V_{P_i} is isomorphic to $\mathcal{E}_i$ as a G-torsor over k?

Here are two cases where the answer is in the affirmative:

(i) G is an abelian group, its 2-primary subgroup is of exponent 2^r, the cyclotomic field extension $k(\mu_{2^r})/k$ is cyclic, and n is arbitrary. This is a special case of [3, Thm. 7.9] (various versions of this statement exist in the literature; see [17], [20]).

(ii) G is arbitrary, k is 'large' and $n = 1$: this is Theorem 1 of the present paper (with the additional piece of information that V may be chosen geometrically integral).

In this appendix, I show by examples that for $n \geq 2$ and k 'large' the answer to the above question is in general in the negative.

In the first part of the appendix, written in April 1999, I consider the case left open in (i) above. I give an example with $G = \mathbf{Z}/8$ and k the 2-adic field $\mathbf{Q}_2$. As may be expected, this example is closely related to Wang's counterexample to Grunwald's theorem.

In the second part of the appendix, written in November 1999, for an arbitrary prime p, I give examples with G a p-group and k a suitable 'large' field. That part builds upon work of Saltman [18].

Background and references for the first part of the appendix (algebraic tori, quasi-trivial and flasque tori, groups of multiplicative type, R-equivalence) will be found in [2], [3], and [21]. For G a commutative algebraic group over a field k, the étale cohomogy group $H^1_{\text{ét}}(k, G)$ may be identified with a Galois cohomology group, and will be simply denoted $H^1(k, G)$.

Proposition A.1. *Let k be a field and A be a finite abelian group. One may embed the constant k-group scheme A into a commutative diagram of exact sequences of k-groups of multiplicative type*:

$$\begin{array}{ccccccccc} 1 & \to & A & \to & P_1 & \to & T & \to & 1 \\ & & \downarrow & & \downarrow & & \downarrow = & & \\ 1 & \to & F & \to & P_2 & \to & T & \to & 1 \end{array}$$

where T is a k-torus, F is a flasque k-torus and P_1 and P_2 are quasi-trivial k-tori.

Proof. By the well-known duality $M \mapsto \hat{M} = \mathrm{Hom}_{k-\mathrm{gr}}(M, \mathbf{G}_{m,k})$ between k-groups of multiplicative type and finitely generated Galois modules over k, it is enough to prove the dual result. There exist exact sequences of finitely generated Galois modules

$$0 \to \hat{T} \to \hat{P}_1 \to \hat{A} \to 0$$

and

$$0 \to \hat{P} \to \hat{F} \to \hat{A} \to 0$$

with $\hat{P}_1$ and $\hat{P}$ permutation modules, and $\hat{F}$ a flasque module (for the second sequence, see [3, (0.6.2)]). The pull-back of the first sequence under the map $\hat{F} \to \hat{A}$ is an exact sequence

$$0 \to \hat{T} \to \hat{P}_2 \to \hat{F} \to 0$$

where the module $\hat{P}_2$ is an extension of the permutation module $\hat{P}_1$ by the permutation module $\hat{P}$, hence is itself a permutation module. Taking duals yields the proposition. $\square$

For a quasi-trivial k-torus P, Hilbert's Theorem 90 implies $H^1(k,P)=0$. Passing over to Galois cohomology in the diagram of Proposition A.1, we get the commutative diagram of exact sequences

$$\begin{array}{ccccccc} P_1(k) & \to & T(k) & \to & H^1(k,A) & \to & 0 \\ \downarrow & & \downarrow = & & \downarrow & & \\ P_2(k) & \to & T(k) & \to & H^1(k,F) & \to & 0. \end{array}$$

From this diagram it immediately follows that the map $H^1(k,A)\to H^1(k,F)$ is onto.

Let us recall the following basic fact from [2]: the map $T(k)\to H^1(k,F)$ induces an isomorphism $T(k)/R \simeq H^1(k,F)$. Here R denotes R-equivalence ([2, §4]) on the set of k-points of the k-torus T.

Proposition A.2. *With notation as above, assume that there exists $\xi \neq 0 \in H^1(k,F)$. Let $\eta \in H^1(k,A)$ denote a lift of ξ under the surjective map $H^1(k,A)\to H^1(k,F)$. Then there do not exist an open set $U\subset \mathbf{A}^1_k$ and an A-torsor $X\to U$ with the following properties: there exist points $M,N\in U(k)$ such that the fibre of $X\to U$ at M is trivial while the fibre of $X\to U$ at N has class $\eta\in H^1(k,A)$.*

Proof. Let us assume there exist such U,M,N. Since P_1 is a quasi-trivial k-torus, for any k-scheme V the étale cohomology group $H^1_{\text{ét}}(V,P_1)$ is isomorphic to a sum of groups $\mathrm{Pic}(V\times_k K_i)$, where the K_i/k are finite separable field extensions of k. For $U\subset \mathbf{A}^1_k$, we thus have $H^1_{\text{ét}}(U,P_1)=0$. Hence the map $T(U)\to H^1_{\text{ét}}(U,A)$ associated to the upper exact sequence in the diagram of Proposition A.1 is onto. There thus exists a k-morphism $\varphi: U\to T$ such that $\varphi^*(P_1\to T)$ is isomorphic to the A-torsor $X\to U$. The map $T(k)\to H^1(k,A)$ sends $\varphi(M)$ to 0, and it sends $\varphi(N)$ to η. Thus the map $T(k)\to H^1(k,F)$ sends $\varphi(M)$ to 0, and it sends $\varphi(N)$ to $\xi\neq 0$. Now since U is an open set of $\mathbf{A}^1_k$, the points $\varphi(M)\in T(k)$ and $\varphi(N)\in T(k)$ are R-equivalent: their images under the map $T(k)\to H^1(k,F)$ should coincide. This contradiction establishes our contention. □

We still need to exhibit one case where the hypotheses of Proposition A.2 are fulfilled. Let k be a field, let $A=\mathbf{Z}/8$ and let T and F be two k-tori as in Proposition A.1. Suppose the cyclotomic field extension $k(\mu_8)/k$ has degree 4. Its Galois group is then $\mathbf{Z}/2\times\mathbf{Z}/2$. In that case, we have $H^1(k,\hat{F})=\mathbf{Z}/2$ ([21, §7.4, p. 79]). If k is a p-adic field, then the finite abelian groups $H^1(k,S)$ and $H^1(k,\hat{S})$ are dual (Tate-Nakayama). Let k be the 2-adic field $\mathbf{Q}_2$. The field extension $\mathbf{Q}_2(\mu_8)/\mathbf{Q}_2$ has degree 4; we thus have $H^1(\mathbf{Q}_2,F)\neq 0$.

This completes the construction of the announced example, but one can be more explicit. Let $k = \mathbf{Q}_2$. As a class $\eta \neq 0 \in H^1(k, \mathbf{Z}/8)$, let us take the class of the degree 8 unramified field extension E of $k = \mathbf{Q}_2$. Let us write the commutative diagram in Proposition A.1 over $\mathbf{Q}$. One may then write the ensuing commutative diagram over $\mathbf{Q}$ and over $\mathbf{Q}_2$, in a compatible manner. Let $M \in T(k)$ be any point with image η in $H^1(k, \mathbf{Z}/8)$. Suppose the image of η in $H^1(k, F)$ is trivial. Then M comes from a k-point of P_2. But then the point M lies in the closure of $T(\mathbf{Q})$ in $T(\mathbf{Q}_2)$, since $P_2/\mathbf{Q}$ is a quasi-trivial torus, hence $\mathbf{Q}$-isomorphic to an open set of some affine space over $\mathbf{Q}$. One can then find a $\mathbf{Q}$-point N of T such that the fibre of $P_1 \to T$ at N is a Galois extension $F/\mathbf{Q}$ with group $\mathbf{Z}/8$ and such that $F \otimes_{\mathbf{Q}} \mathbf{Q}_2 \simeq E$ (as Galois extensions of $\mathbf{Q}_2$ with group $\mathbf{Z}/8$). But there is no such extension (Wang's well-known counterexample to Grunwald's theorem, see [17] and [20]). Thus the image of η in $H^1(k, F)$ is nontrivial.

Let us now turn to other types of examples.

PROPOSITION A.3. *Let p be a prime number. There exist a p-group G, a 'large' field k, and G-torsors $\mathcal{E}_1$ and $\mathcal{E}_2$ over k with the following property: given any G-torsor $f : V \to U$ over an open set U of $\mathbf{A}^1_k$, there do not exist k-points $P, Q \in U(k)$ such that the G-torsor V_P is isomorphic to $\mathcal{E}_1$ and the G-torsor V_Q is isomorphic to $\mathcal{E}_2$.*

Proof. Saltman's work [18] (extended by Bogomolov [1], see [21, §7.6 and §7.7]) produces finite p-groups G together with faithful (finite dimensional) linear representations W of G over the complex field $\mathbf{C}$, such that the unramified Brauer group $\mathrm{Br}_{nr}(F)$ of $F = \mathbf{C}(W)^G$ is a nontrivial (p-primary) group. Here by $\mathbf{C}(W)$ we denote the fraction field of the symmetric algebra on W. The unramified Brauer group of F is the subgroup of the Brauer group $\mathrm{Br}(F)$ consisting of classes which are unramified with respect to any (rank one) discrete valuation on F. As is well-known, the group $\mathrm{Br}_{nr}(\mathbf{C}(W)^G)$ does not depend on the particular faithful (finite dimensional) linear representation of G.

Let us fix one such p-group G. As in the beginning of Section 1, let us fix a homomorphic embedding $G \to \mathrm{GL}_n = \mathrm{GL}_{n,\mathbf{C}}$. We may take for W the vector space of $\mathbf{C}$-points of M_n (the ring scheme of n by n matrices over $\mathbf{C}$), with the action induced by left multiplication. Let $U = \mathrm{GL}_n/G$ and $V = \mathrm{GL}_n \subset M_n$. Projection $V \to U$ makes V into a G-torsor, whose properties are described at the beginning of Section 1.

By Hironaka's theorem, there exists a smooth projective variety $X/\mathbf{C}$ containing U as a dense open set. The function field $\mathbf{C}(X)$ of X is F. By results of Grothendieck, the natural map from the étale Brauer group $\mathrm{Br}(X) = H^2_{\text{ét}}(X, \mathbf{G}_m)$ to $\mathrm{Br}(F)$ is one-to-one, and it induces an isomorphism $\mathrm{Br}(X) \simeq \mathrm{Br}_{nr}(F)$ (see [4]). Let $\mathcal{A} \in \mathrm{Br}(X) \subset \mathrm{Br}(F)$ be a nontrivial element. Let X_F

be the smooth, projective F-variety $X_F = X \times_{\mathbf{C}} F$. This contains the open set $U_F = U \times_{\mathbf{C}} F$. On the one hand, the natural field embedding $\mathbf{C} \subset F$ induces an inclusion $X(\mathbf{C}) \subset X_F(F)$ of the set of $\mathbf{C}$-rational points of X into the set of F-rational points of X_F, and similarly $U(\mathbf{C}) \subset U_F(F)$. Let $P \in U_F(F)$ be an arbitrary point in that subset. On the other hand, the generic point $\mathrm{Spec}(F) \to X$ of X gives rise (via the diagonal map) to an F-rational point Q of Y. Let $\mathcal{A}_F \in \mathrm{Br}(X_F)$ be the inverse image of $\mathcal{A}$ under the projection map $X_F \to X$. Let us evaluate $\mathcal{A}_F$ on the F-rational points P and Q. We have $\mathcal{A}_F(P) = 0 \in \mathrm{Br}(F)$ because $\mathcal{A}_F(P)$ comes from $\mathrm{Br}(\mathbf{C})$. We have $\mathcal{A}_F(Q) \neq 0 \in \mathrm{Br}(F)$ because $\mathcal{A}_F(Q)$ is none other than the image of $\mathcal{A} \in \mathrm{Br}(X)$ under the embedding $\mathrm{Br}(X) \hookrightarrow \mathrm{Br}(F)$. Let k be a field, $F \subset k$, such that the induced map $\mathrm{Br}(F) \to \mathrm{Br}(k)$ is one-to-one. Changing the base field from F to k, we obtain rational points which we still denote P, Q in $X_k(k)$, such that $\mathcal{A}_k(P) = 0$ and $\mathcal{A}_k(Q) \neq 0$ in $\mathrm{Br}(k)$. The points P, Q both lie in $U_k = U \times_{\mathbf{C}} k$. Let $\mathcal{E}_1 = V_P$, respectively $\mathcal{E}_2 = V_Q$, be the G-torsors over k defined as the fibre of the G-torsor $V \to U$ at P, respectively Q. Suppose there exist a G-torsor $Z \to Y$ over an open set $Y \subset \mathbf{A}^1_k$ and two k-points $p, q \in Y(k)$ such that the fibre Z_p, respectively Z_q, is a G-torsor over k isomorphic to $\mathcal{E}_1$, respectively $\mathcal{E}_2$. By the general properties of the G-torsor $V_k \to U_k$ (see beginning of §1) and the fact that $\mathrm{Pic}(Y) = 0$, there exists a k-morphism $r : Y \to U_k$ such that the inverse image of the G-torsor $V_k \to U_k$ under r is isomorphic to the G-torsor $Z \to Y$. Let $P_1 = r(p) \in U(k)$ and $Q_1 = r(q) \in U(k)$. Then V_P and V_{P_1} are isomorphic as G-torsors over k, and similarly V_Q and V_{Q_1}. The general properties of the G-torsor $V \to U$ then imply that there exist $g, h \in \mathrm{GL}_n(k)$ such that $gP_1 = P$ and $hQ_1 = Q$. Since GL_n is an open set of an affine space over k, this implies that the k-points P_1 and P of $U_k(k) \subset X_k(k)$ are R-equivalent. Similarly, Q_1 and Q are R-equivalent. Clearly, P_1 and Q_1 are R-equivalent. Thus P and Q are R-equivalent on the projective k-variety X_k. By Prop. 16 of [2] (p. 213) this implies $\mathcal{A}_k(P) = \mathcal{A}_k(Q)$. But then we cannot have $\mathcal{A}_k(P) = 0$ and $\mathcal{A}_k(Q) \neq 0$.

To complete the proof of Proposition A.3, it remains to notice that the field $k = F((t))$ of formal power series in one variable is a 'large' overfield of F for which the map $\mathrm{Br}(F) \to \mathrm{Br}(k)$ is one-to-one. □

Whether examples as in Proposition A.3 may be exhibited over a p-adic field remains to be seen.

C.N.R.S., U.M.R. 8628, Université de Paris-Sud, Orsay, France
E-mail address: colliot@math.u-psud.fr

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